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# Ion Intensity Measurement in a Plasma Reaction Channel

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## 1 Introduction

THIS Note reports the measurement of negative ions formed when $N_2O$ target molecules react with a flowing high temperature argon plasma in a reaction channel. The experimental arrangement shown in Fig. 1 has been described in detail elsewhere.[1] An argon induction plasma is formed in the discharge chamber with a 0–70 kw, 550 kHz, rf generator and expanded into a sonic reaction channel where the gas temperature is in the range of 2000° to 3000°K. Target gas molecules injected into the reaction channel in concentrations of $10^{-3}$ to $10^{-2}$ mole fraction of the argon are collisionally heated to the local argon gas temperature in the channel. At normal channel pressures (10–80 torr) the target molecules reach thermal equilibrium in less than 1 $\mu$sec and react with the argon plasma during their transit through the reaction channel. Typical transit times in the channel are of the order of 10 $\mu$sec. The transit time is dependent on both the channel length and the gas temperature. The species formed in the channel are "chemically frozen" by a rapid freejet expansion into a low-pressure, high-mass-flow vacuum system resulting in a 300-fold pressure reduction. This results in a five orders of magnitude decrease in the rate of product formation, since the rate of product formation is proportional to the product of the reactant number densities if reaction cross sections are assumed to remain constant.

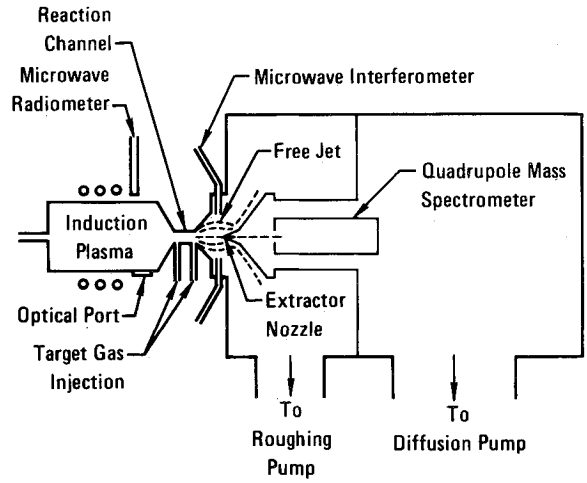


**Fig. 1 Experimental apparatus.**

A $K_a$ band microwave interferometer is used to measure the electron density in the expansion region nine-channel-diameters downstream of the channel exit plane. Electron densities at the interferometer range from $10^{11}$–$10^{13}$ cm$^{-3}$. This corresponds to electron densities in the reaction channel of $3 \times 10^{13}$ to $3 \times 10^{15}$ cm$^{-3}$.

Tungsten—tungsten 75%, rhenium 25%—thermocouples inserted in the reaction channel are used to measure the reaction channel gas temperature. Gas temperatures in the channel have also been measured spectroscopically using radiation from rotational excitation of diatomic molecules.[2]

## 2 Assumptions and Basic Equations

Chemical reactions are assumed to occur only in the reaction channel downstream of the point of target gas injection. Reaction products are sampled from the freejet through a conical extractor into a high vacuum ($10^{-6}$ torr) region for analysis with a quadrupole mass spectrometer.[3] Results of previous investigations of negative ion formation mechanisms with $N_2O$ target molecules at high gas temperatures (2000–3000°K) using this apparatus have shown dissociative electron attachment to be the primary process.[4] For short reaction times the principal product is formed according to

$$N_2O + e \xrightarrow{k_1} O^- + N_2 \tag{1}$$

For longer reaction times, however, additional reactions are necessary to account for the presence of additional negative ions.[3] A reaction sequence which would describe the observed ions when coupled with Eq. (1) is

$$O^- + N_2O \xrightarrow{k_2} NO^- + NO \tag{2}$$

$$O^- + N_2O \xrightarrow{k_3} O_2^- + N_2 \tag{3}$$

$$NO^- + N_2O \xrightarrow{k_4} NO_2^- + N_2 \tag{4}$$

This Note compares the predicted $O^-$ intensities using a binary collision model with the $O^-$ intensities measured mass spectrometrically using three 50-$\mu$m-diam flush-mounted supersonic

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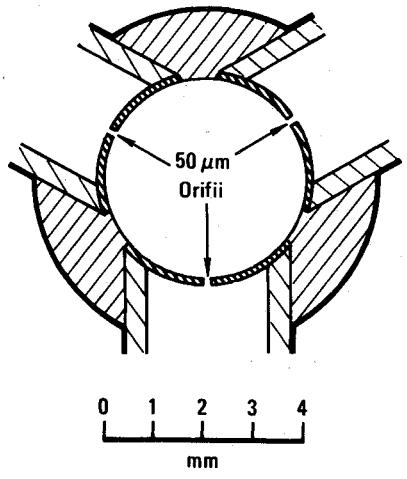


Fig. 2 Flush-mounted supersonic injectors.

target gas injectors shown in Fig. 2. In the reaction model, binary collisions are assumed to be predominantly responsible for the formation of the negative ions. The rate constants are taken to be independent of the reactant and product concentrations, and are time invariant but strongly temperature dependent. In addition, the reactant concentrations are taken to be uniform across the channel and the reaction time proportional to the channel transit time.

The rate equations corresponding to the sequence of reactions for the formation of negative ions are

$$d[O^-]/dt = [N_2O]\{k_1[e]-(k_2+k_3)[O^-]\} \quad (5)$$

$$d[NO^-]/dt = [N_2O]\{k_2[O^-]-k_4[NO^-]\} \quad (6)$$

$$d[O_2^-]/dt = [N_2O][O^-] \quad (7)$$

$$d[NO_2^-]/dt = k_4[N_2O][NO^-] \quad (8)$$

where

$$[N_2O] = [N_2O]_0-[O^-]-[NO^-]-[O_2^-]-[NO_2^-]$$

The three-body collisional radiative recombination theory of Biberman et al.[5] was used to include the ionization and recombination effects of the argon plasma in the reaction channel. This approach allows a wide range of plasma conditions in the reaction region to be considered, including departures from local thermodynamic equilibrium as well as possible departures from Maxwellian velocity distributions of the free electrons. To account for argon recombination and ionization one must include the following additional rate equation

$$d[e]/dt = -\alpha(T_e)[e]^3+\beta(T_e)[Ar][e]-k_1[e][N_2O] \quad (9)$$

The recombination and ionization coefficients $\alpha$ and $\beta$, respectively, are functions of the electron temperature $T_e$ of the plasma. These rate quations, Eqs. (5–9), coupling the background argon plasma and the target molecules, can be solved using numerical methods such as the Runge-Kutta or the Hamming modified predictor-corrector methods.

This model has been applied to reactions of $N_2O$ with an argon plasma for a range of temperatures of 2000° to 3000°K. The calculated $O^-$ formation using this model is given as a family of curves with a parameter **a**. The parameter $\mathbf{a}(=k_1[N_e]_0\tau)$ represents the normalized rate constant for the dissociative electron attachment reaction, expressed in terms of the initial electron density $[N_e]_0$ and the total reaction time $\tau$. Figure 3 shows experimental data along with theoretically predicted values for $\mathbf{a} = 1$, 2, and 5.

Interpretation of the experimental results requires knowledge of the target gas concentration in the channel. The uniformity in target gas concentration at any cross section of the reaction channel depends on the degree of mixing of the target gas with the flowing argon plasma. Target gas molecules are injected into the reaction channel supersonically through orifices contoured to the inside channel to present a smooth wall surface to the flowing plasma. Calculations based on the theory for supersonic injection by Orth and Funk[6] and Billig et al.,[7] indicate that the target gas penetrates the flowing plasma to a depth equal to the channel radius by the time the end of the channel is reached. Furthermore, measurements of ion intensities and electron density as well as visual observations gave no indication of flow separation which has been observed with protruding subsonic injectors.[8]

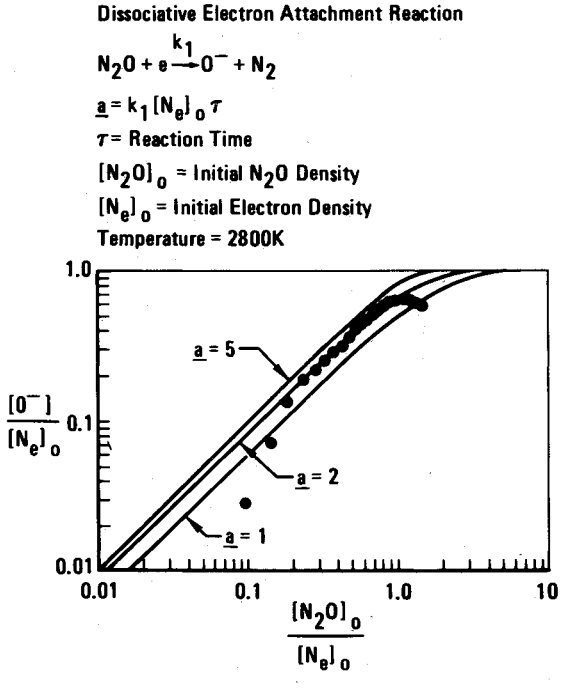


Fig. 3 Comparison of experimental data with theoretically predicted $O^-$ production.

## 3. Conclusions

Target gas injectors that are flush mounted with the reaction channel wall present a smooth boundary to the plasma flow and preserve its uniform characteristics in the channel. Supersonic injection provides good penetration of the target gas into the flowing plasma and results in observed $O^-$ concentrations which agree with the predicted concentrations for a value of the normalized rate constant $\mathbf{a} = 2$ as shown in Fig. 2. The $\mathbf{a} = 2$ curve of Fig. 2 corresponds to an electron attachment rate constant of $1.3 \times 10^{-10}$ cm$^3$ molecule$^{-1}$ sec$^{-1}$.

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# Coupled Thermally Induced Vibrations of Beams

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## Introduction

THE field equations for coupled thermoelastic vibrations of Rayleigh and Timoshenko beams[1] have been derived by Jones.[2] Boley and Barber investigated thermally induced flexural vibration of a simply supported Bernouilli-Euler beam due to heat input to one edge of the span using the one-dimensional heat conduction equation.[3] This Note compares thermally induced vibrations of a simply supported beam with heat input to one edge for Bernouilli-Euler and Timoshenko beams when the field equations with one-dimensional heat conduction are thermoelastically coupled and uncoupled; and for the uncoupled case of two-dimensional heat conduction. The latter analysis shows that a step heat input may cause significant vibrations in short beams as well as long beams.

*The Timoshenko beam: thermoelastically coupled vibrations*

Considering flexural vibrations the basic equations

$$k\oint x\frac{\partial\theta}{\partial n}\,ds+k\tilde{\theta}_{y,zz}-c_0\tilde{\theta}_{y,t}-\frac{kA}{I_{yy}}\tilde{\theta}_y=-\beta_0 TI_{yy}u_{,zzt}+\beta_0\frac{TI_{yy}\rho}{K_x\mu}\cdot u_{,ttt} \tag{1a}$$

$$EI_{yy}u_{,zzzz}+\rho Au_{,tt}-\left(\rho I_{yy}+\frac{EI_{yy}\rho}{K_x\mu}\right)u_{,zztt}+\frac{\rho^2 I_{yy}}{K_x\mu}\cdot u_{,ttt}+\beta_0\tilde{\theta}_{y,zz}=0 \tag{1b}$$

can be obtained from Jones[2] except for the integral

$$k\oint x(\partial\theta/\partial n)\,ds=-qh/2 \tag{2}$$

which accounts for the heat input to one edge of the beam. In Eqs. (1) and (2), a comma followed by a subscript denotes partial differentiation with respect to the subscript, $E$ is Young's modulus, $\alpha$ is the thermal coefficient of linear expansion, $k$ is the thermal conductivity, $\theta$ is the temperature increment above the reference absolute temperature $T$ for the state of zero stress and strain, $I_{yy}$ is the cross-sectional moment of inertia, $A$ is the cross-sectional area, $\rho$ is the density, $K_x$ is the Timoshenko shear coefficient, $t$ is time, $u$ is the transverse deformation, $h$ is the beam depth, $q$ the input heat flux, $s$ is the circumferential arc length along the beam section, $x$ and $y$ are rectangular coordinates in the transverse beam directions, $z$ is the rectangular coordinate in the beam lengthwise direction, and

$$\beta_0=\beta(1-2\nu)=E\alpha,\quad c_0=c(1-\varepsilon)$$
$$\varepsilon=\frac{(3\lambda+2\mu)^2\alpha^2 T}{(\lambda+2\mu)c},\quad \tilde{\theta}_y=\int_A x\theta\,dA \tag{3}$$

where in $\lambda$ and $\mu$ are Lamé's constants, $c$ is the specific heat per unit volume, and $\nu$ is Poisson's ratio.

We assume that $I_{xy}=0$, the beam is of unit width, has pinned ends and is at zero temperature so that the boundary conditions are

$$u(0,t)=u_{,zz}(0,t)=u(l,t)=u_{,zz}(l,t)=\tilde{\theta}_y(0,t)=\tilde{\theta}_y(l,t)=0 \quad \text{for } z=0,l;\ t>0 \tag{4}$$

We assume that the beam is initially at rest with zero initial displacement and temperature so that the initial conditions are

$$u(z,0)=u_{,t}(z,0)=u_{,tt}(z,0)=u_{,ttt}(z,0)=\tilde{\theta}_y(z,0)=0 \quad \text{for } 0\leqq z\leqq l;\ t=0 \tag{5}$$

The solution to the initial-value problem presented by Eqs. (1, 4, and 5) is

$$u(z,t)=\frac{-2\beta_0 qK_x\mu}{\rho^2 I_{yy}c_0}\cdot\frac{h}{l}\sum_{n=1,3,5}^{\infty}\alpha_n(I_1+I_2+I_3)\sin\alpha_n z \tag{6}$$

where

$$I_1=\frac{\omega_{r_1}-e^{-\omega_{i_1}t}(\omega_{i_1}\sin\omega_{r_1}t+\omega_{r_1}\cos\omega_{r_1}t)}{\omega_{r_1}\cdot\omega_{i_5}(\omega_{i_1}^2+\omega_{r_1}^2)(\omega_{i_3}^2+\omega_{r_3}^2)}$$

$$I_2=-\left\{\frac{e^{-\omega_{i_5}t}\omega_{r_1}-e^{-\omega_{i_1}t}[(\omega_{i_1}-\omega_{i_5})\sin\omega_{r_1}t+\omega_{r_1}\cos\omega_{r_1}t]}{\omega_{r_1}\omega_{i_5}[(\omega_{i_1}-\omega_{i_5})^2+(\omega_{r_1})^2][(\omega_{i_3}-\omega_{i_5})^2+(\omega_{r_3})^2]}\right\} \tag{7}$$

$$I_3=\frac{1}{\omega_{r_1}\omega_{r_3}\{(\omega_{i_3}^2+\omega_{r_3}^2)[(\omega_{i_3}-\omega_{i_5})^2+(\omega_{r_3})^2]\}^{1/2}}\times\Bigg[\sin(\omega_{r_3}t+\phi)\times\Bigg\{\frac{e^{-\omega_{i_1}t}[-(\omega_{i_1}-\omega_{i_3})\sin(\omega_{r_1}-\omega_{r_3})t-(\omega_{r_1}-\omega_{r_3})\times\cos(\omega_{r_1}-\omega_{r_3})t]+(\omega_{r_1}-\omega_{r_3})e^{-\omega_{i_3}t}}{2[(\omega_{i_1}-\omega_{i_3})^2+(\omega_{r_1}-\omega_{r_3})^2]}+\frac{e^{-\omega_{i_1}t}[-(\omega_{i_1}-\omega_{i_3})\sin(\omega_{r_1}+\omega_{r_3})t-(\omega_{r_1}+\omega_{r_3})\times\cos(\omega_{r_1}+\omega_{r_3})t]+(\omega_{r_1}+\omega_{r_3})e^{-\omega_{i_3}t}}{2[(\omega_{i_1}-\omega_{i_3})^2+(\omega_{r_1}+\omega_{r_3})^2]}\Bigg\}-\cos(\omega_{r_3}t+\phi)\times\Bigg\{\frac{e^{-\omega_{i_1}t}[(\omega_{r_1}-\omega_{r_3})\sin(\omega_{r_1}-\omega_{r_3})t-(\omega_{i_1}-\omega_{i_3})\times\cos(\omega_{r_1}-\omega_{r_3})t]+(\omega_{i_1}-\omega_{i_3})e^{-\omega_{i_3}t}}{2[(\omega_{i_1}-\omega_{i_3})^2+(\omega_{r_1}-\omega_{r_3})^2]}-\frac{e^{-\omega_{i_1}t}[(\omega_{r_1}+\omega_{r_3})\sin(\omega_{r_1}+\omega_{r_3})t-(\omega_{i_1}-\omega_{i_3})\times\cos(\omega_{r_1}+\omega_{r_3})t]+(\omega_{i_1}-\omega_{i_3})e^{-\omega_{i_3}t}}{2[(\omega_{i_1}-\omega_{i_3})^2+(\omega_{r_1}+\omega_{r_3})^2]}\Bigg\}\Bigg]$$

The transverse deflection has a final value

$$\lim_{t\to\infty}u(z,t)=-\frac{h\beta_0 k_x\mu q}{\rho^2 I_{yy}c_0}\cdot\frac{2}{l}\sum_{n=1,3,5}^{\infty}\times\alpha_n\left\{\frac{1}{\omega_{i_5}(\omega_{i_1}^2+\omega_{r_1}^2)(\omega_{i_3}^2+\omega_{r_3}^2)}\right\}\sin\alpha_n z \tag{8}$$

In Eqs. (7) and (8) the roots of the secular equation to the first order of $\varepsilon$ are[1]

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